Ordinary Differentiation of vectors:

Let $\vec{F}(t)$ be a vector ~~field~~ valued function. Then the ordinary derivative of the vector $\vec{F}(t)$ is defined as

$$\frac{d\vec{F}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}$$

Ex: Let $\vec{F}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, where x, y, z ~~are~~ are differentiable function of a scalar t. Prove that

$$\frac{d\vec{F}(t)}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Proof: $$\frac{d\vec{F}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}$$

$$\cancel{= \frac{\{x(t+\Delta t)\hat{i} + y(t+\Delta t)\hat{j} + z(t+\Delta t)\hat{k}\} - \{x\hat{i} + y\hat{j} + z\hat{k}\}}{\Delta t}}$$

$$= \lim_{\Delta t \to 0} \frac{\{x(t+\Delta t)\hat{i} + y(t+\Delta t)\hat{j} + z(t+\Delta t)\hat{k}\} - \{x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}\}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left[\frac{x(t+\Delta t) - x(t)}{\Delta t}\hat{i} + \frac{y(t+\Delta t) - y(t)}{\Delta t}\hat{j} + \frac{z(t+\Delta t) - z(t)}{\Delta t}\hat{k} \right\}$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Ex: Given that $\vec{F}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$

find (i) $\frac{d\vec{F}}{dt}$ (ii) $\frac{d^2\vec{F}}{dt^2}$ (iii) $\frac{d^3F}{dt^3}$ (iv) $\left|\frac{d\vec{F}}{dt}\right|$

(v) $\left|\frac{d^2F}{dt^2}\right|$

(i) $\frac{d\vec{F}(t)}{dt} = \frac{d}{dt}(\cos t)\hat{i} + \frac{d}{dt}(\sin t)\hat{j} + \frac{d}{dt}(t)\hat{k}$

$= -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$

(ii) $\frac{d^2\vec{F}}{dt^2} = -\frac{d}{dt}(\sin t)\hat{i} + \frac{d}{dt}(\cos t)\hat{j} + \frac{d}{dt}(1)\hat{k}$

$= \cancel{-\cos t\,\hat{i} - \sin t\,\hat{j}}$

$= -\cos t\,\hat{i} - \sin t\,\hat{j}$

iii) $\frac{d^3\vec{F}}{dt^3} = -\frac{d}{dt}(\cos t)\hat{i} - \frac{d}{dt}(\sin t)\hat{j} = \sin t\,\hat{i} - \cos t\,\hat{j}$

$\cancel{\left|\frac{d\vec{F}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}}$

$\cancel{\left|\frac{d^2\vec{F}}{dt^2}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1}$

(iv) $\left|\frac{d\vec{F}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{1+1}$

$= \sqrt{2}$

(v) $\left|\frac{d^2\vec{F}}{dt^2}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2 + 0} = \sqrt{\sin^2 t + \cos^2 t}$

$= 1$